Computations on Modular Jacobian Surfaces

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Abstract. We give a method for finding rational equations of genus 2 curves whose jacobians are abelian varieties A_f attached by Shimura to normalized newforms $f \in S_2(\Gamma_0(N))$. We present all the curves corresponding to principally polarized surfaces A_f for $N \leq 500$.

1 Introduction

Given a normalized newform $f = \sum_{n>0} a_n q^n \in S_2(\Gamma_0(N))$, Shimura [5]-[6] attaches to it an abelian variety A_f defined over $\mathbb{Q}$ of dimension equal to the degree of the number field $E_f = \mathbb{Q}(\{a_n\})$. The Eichler-Shimura congruence makes it possible to compute at every prime $p \nmid N$ the characteristic polynomial of the Frobenius endomorphism acting on the Tate module of $A_f/\mathbb{F}_p$ from the coefficient a_p and its Galois conjugates. In consequence, when A_f is $\mathbb{Q}$-isogenous to the jacobian of a curve C defined over $\mathbb{Q}$, the number of points of the reduction of this curve mod a prime p of good reduction can be obtained from the characteristic polynomial of the Hecke operator T_p acting on $H^0(A_f, \Omega^1)$. Among these *jacobian-modular curves*, those which are hyperelliptic of low genus are especially interesting for public key cryptography.

As an optimal quotient of the jacobian of $X_0(N)$, $J_0(N)$, the abelian variety A_f has a natural polarization induced from $J_0(N)$. We will focus our attention on polarized surfaces A_f which are $\mathbb{Q}$-isomorphic to jacobians of genus 2 curves. Wang [7] gave a first step in the determinations of such curves. More precisely, using modular symbols he computed the periods of f and its Galois conjugate and presented A_f as a complex torus with an explicit polarization. For those principally polarized A_f, Wang computed numerically Igusa invariants by means of even Thetanullwerte and built an hyperelliptic curve $C/\mathbb{Q}$ such that $\text{Jac } C \simeq A_f$ over $\overline{\mathbb{Q}}$. The curves C obtained with this procedure have two drawbacks: they have huge coefficients, and, moreover, we only know that their jacobians

are $\overline{\mathbb{Q}}$-isomorphic to the corresponding abelian varieties A_f, but we don't know whether they are $\mathbb{Q}$-isomorphic, or even $\mathbb{Q}$-isogenous. Frey and Muller [2] looked for a curve $C'/\mathbb{Q}$ among the twisted curves of C such that the local factors of the L-series of $\operatorname{Jac} C'$ and A_f agree for all primes less than a large enough bound.

In this paper we want to go one step further in the determination of these jacobian modular surfaces. We describe a more arithmetical and efficient method, based on odd Thetanullwerte, which solves the problem up to numerical approximations. Our method provides equations $C_F : y^2 = F(x)$ with $F(x) \in \mathbb{Q}[x]$ such that $\operatorname{Jac} C_F$ or $\operatorname{Jac} C_{-F}$ is A_f. The sign is chosen using the Eichler-Shimura congruence.

We have implemented a program in MAGMA to determine modular jacobian surfaces and equations for the corresponding curves. We have found all the modular jacobian surfaces of level $N \leq 500$. The equations obtained for the corresponding curves are presented at the end of the paper. It is remarkable that almost all of them are minimal equations over $\mathbb{Z}[1/2]$.

2   Theoretical Foundations

A polarized abelian variety (A, Θ) of dimension g defined over $\mathbb{C}$ can be realized as a complex torus $T_A = \mathbb{C}^g/\Lambda$, where Λ is the period lattice of A with respect to a basis of $H^0(A, \Omega^1)$, with a nondegenerate Riemann form defined on Λ. We choose a symplectic basis for Λ, and write it as a $2g \times g$ matrix $\Omega = (\Omega_1|\Omega_2)$. The normalized period matrix $Z = \Omega_1^{-1}\Omega_2$ satisfies the Riemann conditions $Z = {}^tZ$, $Y = \operatorname{Im} Z$ is positive definite and the Riemann theta function:

$$\theta(z) := \theta(z; Z) := \sum_{n \in \mathbb{Z}^g} \exp(\pi i\, {}^t n . Z . n + 2\pi i\, {}^t n . z)$$

is holomorphic in $\mathbb{C}^g$. The values of the Riemann theta function at 2-torsion points are called Thetanullwerte. Historically, only the even Thetanullwerte, i.e., the values of the theta function at even 2-torsion points have been studied, since the values at odd 2-torsion points are always zero. Anyway, the values of the derivatives of the theta function at the odd 2-torsion points have nice properties, and also do provide useful geometrical information ([4]).

We now give the theoretical results which allow one to recognize when a principally polarized abelian surface is the jacobian of a genus 2 curve.

Proposition 1. *Let (A, Θ) be an irreducible principally polarized abelian surface defined over a number field K. There exists a hyperelliptic curve C of genus 2 defined over K such that $A = \operatorname{Jac} C$.*

Proof: It is well known that the irreducibility of A implies that $A = \operatorname{Jac} C$ for a certain hyperelliptic curve C defined over $\mathbb{C}$. But for genus 2 curves, the Abel-Jacobi map in degree 1 is an isomorphism between the curve C and the Θ divisor in $\operatorname{Jac} C = A$. Hence, we can assume that $C = \Theta$, which is defined over K. $\square$

Proposition 2. *A principally polarized abelian surface (A, Θ) is not irreducible if and only if there is an even 2-torsion point P such that the corresponding even Thetanullwerte vanishes.*

Proof: If (A, Θ) is irreducible principally polarized, then it is isomorphic to the jacobian of a hyperelliptic genus 2 curve, and hence every even Thetanullwerte is non-zero.

Conversely, assume that (A, Θ) is the product of two elliptic curves E_1, E_2. This means that the theta function θ_A associated to the pair (A, Θ) is equal to $\theta_1 \theta_2$, where we denote by θ_i the theta function associated to the elliptic curve E_i. Let O_i be the zero point in E_i, which is the unique odd 2-torsion point in E_i. The pair $O = (O_1, O_2) \in E_1 \times E_2$ gives an even two torsion point in A, which satisfies $\theta_A(O) = 0$.   □

Once we know that a principally polarized abelian surface A is a jacobian, we want a method to find a curve C such that $A \simeq \mathrm{Jac}\, C$. We would like to be careful enough to assure that, when A is defined over a number field K, the curve C and the isomorphism $A \simeq \mathrm{Jac}\, C$ are also defined over K. The following result, which can be found in [4], will be basic for our purpose.

Theorem 1. *Let $F(X) = a_6 X^6 + a_5 X^5 + \ldots + a_1 X + a_0 \in \mathbb{C}[X]$ be a separable polynomial of degree 5 or 6. Let $\Omega = (\Omega_1 | \Omega_2)$ be the period matrix of the hyperelliptic curve $C_F : y^2 = F(x)$ with respect to the basis $\omega_1 = \dfrac{dx}{y}$, $\omega_2 = \dfrac{xdx}{y}$ of $H^0(C_F, \Omega^1)$ and any symplectic basis of $H_1(C_F, \mathbb{Z})$, and take $Z_F = \Omega_1^{-1} \Omega_2$.*

a) *The roots α_k of the polynomial F are the ratios $\dfrac{x_{k,2}}{x_{k,1}}$, given by the solutions $(x_{k,1}, x_{k,2})$ of the six homogeneous linear equations*

$$\left(\frac{\partial \theta}{\partial z_1}(w_k) \quad \frac{\partial \theta}{\partial z_2}(w_k) \right) \Omega_1^{-1} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = 0,$$

where $w_1, \ldots, w_6$ are the six odd 2-torsion points of $J(C_F)$, given by

$$w_1 = \tfrac{1}{2} Z_F \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad w_2 = \tfrac{1}{2} Z_F \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$w_3 = \tfrac{1}{2} Z_F \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad w_4 = \tfrac{1}{2} Z_F \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$w_5 = \tfrac{1}{2} Z_F \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad w_6 = \tfrac{1}{2} Z_F \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

When $\deg F = 5$, one of these ratios is infinity and we discard it.

b) *Let $W_j = (\alpha_j, 0)$ be the Weierstrass point corresponding to w_j. Denote by $H[W_j]$ the hyperplane of $\mathbb{P}^1$ given by the equation*

$$H[W_j](X_1, X_2) := \left(\frac{\partial \theta}{\partial z_1}(w_j) \quad \frac{\partial \theta}{\partial z_2}(w_j) \right) \Omega_1^{-1} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

The discriminant $\Delta_{alg}(C_F)$ of the polynomial F satisfies the relation

$$\Delta_{alg}(C_F)^7 = 2^{120} a_6^{10} \pi^{60} \det \Omega_1^{-30} \prod_{j<k} H[W_j](1, \alpha_k)^2 \text{ if } \deg(F) = 6;$$

$$\Delta_{alg}(C_F)^5 = 2^{80} a_5^{10} \pi^{80} \det \Omega_1^{-20} \prod_{j<k} H[W_j](1, \alpha_k)^2 \text{ if } \deg(F) = 5.$$

3   Determination of Hyperelliptic Equations

We explain here how one can, given an irreducible abelian surface (A, Θ) defined over K, look for a hyperelliptic curve $C_F : Y^2 = F(X)$ such that A is K-isomorphic to $\mathrm{Jac}\, C_F$. We have divided our method into four steps.

Step 1: Period matrix. The first step consists in choosing a suitable period matrix Ω for A. We have to fix a symplectic basis of $H_1(A, \mathbb{Z})$, a convenient basis of $H^0(A, \Omega^1_{A/K})$ and compute the corresponding period matrix. The following result assures us that the basis of regular differentials can be chosen arbitrarily.

Proposition 3. *([3]). Let C/K be a genus 2 curve. For every linearly independent pair of regular differentials $\omega_1, \omega_2 \in H^0(C, \Omega^1_{C/K})$, there exists a polynomial $F(X) \in K[X]$ of degree 5 or 6 without double roots such that the functions on C given by*

$$x = \frac{\omega_1}{\omega_2}, \quad y = \frac{dx}{\omega_2}$$

satisfy the equation $y^2 = F(x)$.

Step 2: Weierstrass points. In this step, we compute the roots α_k of the polynomial F given by the first part of the theorem 1, and we take the monic polynomial $F_0(X) = \prod_k (X - \alpha_k) \in K[X]$.

Step 3: Leading coefficient. With the formulas given for the discriminant in part *b)* of theorem 1, we obtain $a_6^{10} \in K$ (or $a_5^{10} \in K$ if $\deg F_0 = 5$). We choose one of the tenth roots $a_6' \in K$ of this value and take the polynomial $F_1(X) = a_6' F_0(X) \in K[X]$.

Step 4: Hyperelliptic equation. At this point, it only remains to find the tenth root of unity ζ such that $F = \zeta F_1$. Since the curves C_F and $C_{\lambda^2 F}$ with $\lambda \in K^*$ are K-isomorphic, it suffices to consider only the cases $\zeta = 1$ and $\zeta = -1$, when $-1 \notin K^2$. First we check whether C_F and C_{-F} are K-isomorphic. If they are not, then we look if $\mathrm{Jac}\, C_F$ and $\mathrm{Jac}\, C_{-F}$ are not K-isogenous. In this case, by Faltings Theorem, only one of their L-series will agree with the L-series of A and this will give the right sign for $F = \pm F_1$. In fact, it will suffice to find a prime $\mathfrak{p}$ in K of good reduction for the curves C_F and C_{-F} such that their reductions mod $\mathfrak{p}$ have a different number of points.

In the case that C_F and C_F are not K-isomorphic and $\mathrm{Jac}\, C_F$ and $\mathrm{Jac}\, C_{-F}$ are K-isogenous, we cannot determine the right sign. Anyway, we know that both jacobians $\mathrm{Jac}\, C_F$ and $\mathrm{Jac}\, C_{-F}$ are $K(\sqrt{-1})$-isomorphic to A_f, and one of them is K-isomorphic.

4 Modular Computations

We apply the method described in the previous section to present the irreducible principally polarized two-dimensional factors of $J_0(N)^{\text{new}}$ as jacobians of curves, for $N \leq 500$.

In order to do this, we begin looking for the normalized newforms $f = \sum a_n q^n \in S_2(\Gamma_0(N))$ such that the number field $E_f = \mathbb{Q}(\{a_n\})$ is quadratic. For each of these newforms, we take an integral basis of the $\mathbb{C}$-vector space generated by f and its Galois conjugate ${}^\sigma f$. We also determine a symplectic basis of $H_1(A_f, \mathbb{Z})$. If A_f is principally polarized, we compute the period matrix with respect to these bases, using the package on modular symbols written by W. Stein in `Magma`.

Next, we check the irreducibility of A_f by means of proposition 2. We remark that all the A_f studied are irreducible.

We now apply the method of section 3. We follow the steps described there, to find the corresponding curves $C_F : Y^2 = F(X)$. Since we are working over $\mathbb{Q}$, we can change the polynomial $F(X)$ in order to obtain an integral equation. We multiply $F(X)$ by $d = t/b$, where $t \in \mathbb{Z}$ is the square of the l.c.m. of the denominators of the coefficients of F, and $b \in \mathbb{Z}$ is the g.c.d. of their numerators divided by its maximum square-free factor. It is worth remarking that the equations obtained have very small coefficients, even before finding the integral model.

The only case in which we have found a curve C_F such that $\operatorname{Jac} C_F$ and $\operatorname{Jac} C_{-F}$ are $\mathbb{Q}$-isogenous occurs for $N = 256$, but in fact both curves are already $\mathbb{Q}$-isomorphic, because the corresponding polynomial $F(X)$ is odd.

We have used three tests to check the correctness of our equations. First, we have computed the absolute Igusa invariants of the curves C_F in two different ways: algebraically from the coefficients of our equations, and numerically from the even Thetanullwerte of the period matrix. They have agreed to high accuracy in all cases. Second, we have compared the local factors of the L-series of $\operatorname{Jac} C_F$ and A_f for all primes $p < 100$ not dividing $\Delta_{alg}(C_F)$. Finally, we have computed the odd part of the conductor of C_F using the program `genus2reduction` by Q. Liu. In all cases, this odd part agreed with the odd part of the square of the level of the newform f, as it should by [1]. It is worth noting that in almost all cases our equations are minimal over $\mathbb{Z}[1/2]$.

We illustrate our computations with an example. The first level for which $J_0(N)^{\text{new}}$ has a proper two-dimensional factor is $N = 63$. Using `Magma` we identify the corresponding normalized newform f:

$$f = q + \sqrt{3}q^2 + q^4 - 2\sqrt{3}q^5 + q^7 - \sqrt{3}q^8 - 6q^{10} + 2\sqrt{3}q^{11} + 2q^{13} + \ldots$$

An integral basis of the space $\langle f, {}^\sigma f \rangle$ is

$$f_1 = q + q^4 + q^7 - 6q^{10} + 2q^{13} + \ldots, \qquad\qquad f_2 = q^2 - 2q^5 - q^8 + 2q^{11} + \ldots$$

A basis for $H_1(A_f, \mathbb{Z})$ in terms of modular symbols is given by

$$\gamma_1 = \{-\tfrac{1}{24}, 0\} - \{-\tfrac{1}{28}, 0\} + \{-\tfrac{1}{30}, 0\} - \{-\tfrac{1}{51}, 0\} - \{-\tfrac{1}{3}, -\tfrac{2}{7}\},$$

$$\gamma_2 = \{-\tfrac{1}{24}, 0\} - \{-\tfrac{1}{28}, 0\} + \{-\tfrac{1}{39}, 0\} - \{-\tfrac{1}{57}, 0\} - \{-\tfrac{1}{6}, -\tfrac{1}{7}\},$$

$$\gamma_3 = \{-\tfrac{1}{24}, 0\} + \{-\tfrac{1}{39}, 0\} - \{-\tfrac{1}{45}, 0\} - \{-\tfrac{1}{60}, 0\} - \{-\tfrac{1}{3}, -\tfrac{2}{7}\} - \{\tfrac{3}{7}, \tfrac{4}{9}\},$$

$$\gamma_4 = \{-\tfrac{1}{36}, 0\} - \{-\tfrac{1}{49}, 0\} + \{-\tfrac{1}{51}, 0\} - \{-\tfrac{1}{54}, 0\} + \{-\tfrac{1}{57}, 0\}$$

$$-\{-\tfrac{1}{60}, 0\} - \{-\tfrac{1}{3}, -\tfrac{2}{7}\}.$$

Computing the intersection matrix of these paths we see that A_f is principally polarized. We find a symplectic basis for $H_1(A_f, \mathbb{Z})$, and compute the periods of f_1, f_2 with respect to these bases. We obtain as period matrix $\Omega = (\Omega_1 \mid \Omega_2)$ for A_f:

$$\Omega_1 = \begin{pmatrix} 0.3590439\ldots + i * 0.6218823\ldots & -2.2150442\ldots + i * 1.2788564\ldots \\ -2.2150442\ldots + i * 3.8365691\ldots & 1.0771318\ldots + i * 0.6218823\ldots \end{pmatrix},$$

$$\Omega_2 = \begin{pmatrix} -1.4969563\ldots + i * 1.2788564\ldots & -1.8560003\ldots - i * 0.6569740\ldots \\ -3.3529566\ldots + i * 0.6218823\ldots & -1.1379124\ldots + i * 3.2146868\ldots \end{pmatrix}.$$

We apply the method described in section 3, to obtain the monic polynomial

$$F_0(x) = x^6 - 54x^3 - 27.$$

The coefficient a_6 is $1/12$, so that $F_1(x) = 1/12 F_0(x)$. The first prime for which the local factors of C_{F_1} and C_{-F_1} are different is $p = 67$. Comparing with the polynomial

$$x^2(x + p/x - a_p)(x + p/x - {}^\sigma a_p),$$

we see that the right sign is -1. We multiply $-F_1(x)$ by 6^2 to obtain an integral equation. We can finally assert that A_f is the jacobian of the curve

$$y^2 = -3x^6 + 162x^3 + 81.$$

The Igusa invariants of this curve are

$$i_1 = \frac{2^3 \cdot 37^5}{3 \cdot 7^3}, \qquad i_2 = -\frac{3 \cdot 37^3 \cdot 103}{2 \cdot 7^3} \qquad i_3 = -\frac{5 \cdot 37^2 \cdot 881}{2^3 \cdot 7^3}.$$

We have also computed these Igusa invariants from the even Thetanullwerte associated to the period matrix Z, obtaining, of course, the same result.

Using Q. Liu's program, we find a minimal equation for the curve C:

$$Y^2 = X^6 + 54X^3 - 27,$$

which is obtained from our equation through the change $x = 3/X$, $y = 9Y/X^3$, which corresponds essentially to a different ordering of the modular forms f_1, f_2 as basis of $\langle f, {}^\sigma f \rangle$.

5   Tables

We present the equations that we have obtained in the following table. We have labelled the irreducible principally polarized two-dimensional factors A_f of $J_0(N)^{\text{new}}$ as S_{NX}. We have ordered the two-dimensional factors of $J_0(N)^{\text{new}}$ following the output of the Magma function SortDecomposition. The letter X denotes the position of A_f with respect to this ordering. The third column indicates when we know that the given equation is minimal over $\mathbb{Z}[1/2]$.

A_f	$C_F : y^2 = F(x), \quad \text{Jac}\, C_F \simeq A_f$	minimal?
S_{23A}	$y^2 = x^6 - 8x^5 + 2x^4 + 2x^3 - 11x^2 + 10x - 7$	yes
S_{29A}	$y^2 = x^6 - 4x^5 - 12x^4 + 2x^3 + 8x^2 + 8x - 7$	yes
S_{31A}	$y^2 = x^6 - 8x^5 + 6x^4 + 18x^3 - 11x^2 - 14x - 3$	yes
S_{63B}	$y^2 = -3x^6 + 162x^3 + 81$	
S_{65B}	$y^2 = -x^6 - 4x^5 + 3x^4 + 28x^3 - 7x^2 - 62x + 42$	yes
S_{65C}	$y^2 = -15x^6 + 36x^4 - 30x^3 + 72x^2 - 39$	yes
S_{67B}	$y^2 = x^6 + 2x^5 + x^4 - 2x^3 + 2x^2 - 4x + 1$	yes
S_{73B}	$y^2 = x^6 - 4x^5 + 2x^4 + 6x^3 + x^2 + 2x + 1$	yes
S_{87A}	$y^2 = x^6 - 2x^4 - 6x^3 - 11x^2 - 6x - 3$	yes
S_{93A}	$y^2 = x^6 + 2x^4 - 6x^3 + 5x^2 + 6x + 1$	yes
S_{103A}	$y^2 = x^6 + 2x^4 + 2x^3 + 5x^2 + 6x + 1$	yes
S_{107A}	$y^2 = x^6 + 2x^5 + 5x^4 + 2x^3 - 2x^2 - 4x - 3$	yes
S_{115B}	$y^2 = x^6 + 2x^4 + 10x^3 + 5x^2 + 6x + 1$	yes
S_{117B}	$y^2 = x^6 - 10x^3 - 27$	yes
S_{117C}	$y^2 = -3x^6 - 12x^4 - 18x^3 - 48x^2 - 36x - 27$	yes
S_{125A}	$y^2 = x^6 + 2x^5 + 5x^4 + 10x^3 + 10x^2 + 8x + 1$	yes
S_{125B}	$y^2 = 5x^6 - 10x^5 + 25x^4 - 50x^3 + 50x^2 - 40x + 5$	yes
S_{133A}	$y^2 = x^6 - 2x^5 + 5x^4 - 6x^3 + 10x^2 - 8x + 1$	yes
S_{133B}	$y^2 = -3x^6 - 22x^5 - 35x^4 + 50x^3 + 74x^2 - 100x + 29$	yes
S_{135D}	$y^2 = x^6 + 6x^4 - 10x^3 + 9x^2 - 30x - 11$	yes
S_{147D}	$y^2 = x^6 - 4x^4 + 2x^3 + 8x^2 - 12x + 9$	yes
S_{161B}	$y^2 = x^6 + 6x^5 + 17x^4 + 22x^3 + 26x^2 + 12x + 1$	yes
S_{167A}	$y^2 = x^6 - 4x^5 + 2x^4 - 2x^3 - 3x^2 + 2x - 3$	yes
S_{175E}	$y^2 = x^6 + 2x^5 - 3x^4 + 6x^3 - 14x^2 + 8x - 3$	yes
S_{177A}	$y^2 = x^6 + 2x^4 - 6x^3 + 5x^2 - 6x + 1$	yes
S_{177B}	$y^2 = -15x^6 - 120x^5 - 530x^4 - 710x^3 - 515x^2 - 30x + 45$	
S_{188B}	$y^2 = x^5 - x^4 + x^3 + x^2 - 2x + 1$	yes
S_{189E}	$y^2 = x^6 - 2x^3 - 27$	yes

A_f	$C_F : y^2 = F(x)$, $\operatorname{Jac} C_F \simeq A_f$	minimal?
S_{191A}	$y^2 = x^6 + 2x^4 + 2x^3 + 5x^2 - 6x + 1$	yes
S_{205D}	$y^2 = x^6 + 2x^4 + 10x^3 + 5x^2 - 6x + 1$	yes
S_{209B}	$y^2 = x^6 - 4x^5 + 8x^4 - 8x^3 + 8x^2 + 4x + 4$	yes
S_{213B}	$y^2 = x^6 + 2x^4 + 2x^3 - 7x^2 + 6x - 3$	yes
S_{221C}	$y^2 = x^6 - 2x^5 + x^4 + 6x^3 + 2x^2 + 4x + 1$	yes
S_{224C}	$y^2 = -2x^6 - 8x^5 - 34x^4 - 48x^3 - 118x^2 + 56x + 154$	yes
S_{224D}	$y^2 = 2x^6 - 8x^5 + 34x^4 - 48x^3 + 118x^2 + 56x - 154$	yes
S_{243C}	$y^2 = x^6 + 6x^3 - 27$	yes
S_{250D}	$y^2 = 20\,x^6 - 140\,x^5 + 325\,x^4 + 1050\,x^3 + 425\,x^2 + 160\,x + 80$	
S_{256E}	$y^2 = 2\,x^5 - 128\,x$	yes
S_{261A}	$y^2 = x^6 - 6x^4 + 10x^3 + 21x^2 - 30x + 9$	yes
S_{261B}	$y^2 = -3x^6 + 18x^4 + 30x^3 - 63x^2 - 90x - 27$	yes
S_{261D}	$y^2 = -3x^6 + 6x^4 - 18x^3 + 33x^2 - 18x + 9$	yes
S_{262C}	$y^2 = -8x^5 + 56x^4 - 82x^3 - 312x^2 - 264x - 64$	yes
S_{266B}	$y^2 = 8\,x^6 + 16\,x^5 + 13\,x^4 + 6\,x^3 - 19\,x^2 - 8\,x - 16$	yes
S_{268C}	$y^2 = x^6 - 2x^5 + x^4 - 4x^3 + 2x^2 + 4x + 1$	yes
S_{275G}	$y^2 = -3x^6 - 2x^5 + x^4 - 14x^3 + 2x^2 - 8x + 1$	yes
S_{279A}	$y^2 = -3\,x^6 - 6\,x^4 - 18\,x^3 - 15\,x^2 + 18\,x - 3$	yes
S_{279B}	$y^2 = -3\,x^6 + 6\,x^5 - 3\,x^4 - 6\,x^3 + 18\,x^2 - 12\,x + 9$	yes
S_{287A}	$y^2 = x^6 + 2x^5 - 3x^4 - 6x^3 - 10x^2 - 4x - 3$	yes
S_{292A}	$y^2 = -x^6 - 2x^5 - 4x^4 - 4x^3 - 3x^2 - 2x + 1$	yes
S_{297E}	$y^2 = x^6 - 12\,x^4 - 8\,x^3 + 12\,x^2 - 12\,x + 4$	yes
S_{297F}	$y^2 = -3\,x^6 + 36\,x^4 - 24\,x^3 - 36\,x^2 - 36\,x - 12$	yes
S_{299A}	$y^2 = -3x^6 - 10x^5 - 7x^4 + 6x^3 + 6x^2 - 4x + 1$	yes
S_{325H}	$y^2 = -75\,x^6 + 180\,x^4 + 150\,x^3 + 360\,x^2 - 195$	yes
S_{335B}	$y^2 = x^6 - 4x^5 - 48x^2 - 20x - 4$	yes
S_{345G}	$y^2 = x^6 - 12\,x^5 + 32\,x^4 + 24\,x^3 + 8\,x^2 - 12\,x + 4$	yes
S_{351A}	$y^2 = x^6 - 6x^4 + 18x^3 + 9x^2 - 18x + 5$	yes
S_{351C}	$y^2 = -3\,x^6 + 18\,x^4 + 54\,x^3 - 27\,x^2 - 54\,x - 15$	yes
S_{351D}	$y^2 = 21\,x^6 - 210\,x^5 + 525\,x^4 - 602\,x^3 + 714\,x^2 + 336\,x + 665$	
S_{357E}	$y^2 = x^6 + 8x^4 - 8x^3 + 20x^2 - 12x + 12$	yes
S_{375C}	$y^2 = 105\,x^6 + 240\,x^5 + 550\,x^4 + 450\,x^3 + 325\,x^2 + 90\,x - 155$	yes
S_{376A}	$y^2 = -x^5 - x^4 + 3\,x^3 + 3\,x^2 - 4\,x + 1$	yes
S_{376B}	$y^2 = x^5 - x^3 + 2x^2 - 2x + 1$	yes
S_{380D}	$y^2 = x^5 - 7x^3 - 4x^2 + 5x + 5$	yes

A_f	$C_F : y^2 = F(x), \quad \mathrm{Jac}\, C_F \simeq A_f$	minimal?
S_{387F}	$y^2 = -12x^6 + 162x^3 + 324$	
S_{389B}	$y^2 = x^6 + 10x^5 + 23x^4 - 20x^3 - 45x^2 + 46x - 11$	yes
S_{391A}	$y^2 = x^6 + 10x^4 - 6x^3 - 11x^2 + 18x - 7$	yes
S_{424A}	$y^2 = x^6 - 2x^5 + 6x^4 - 8x^3 + 10x^2 - 8x + 5$	yes
S_{440E}	$y^2 = x^5 + 2x^3 - 11x^2 - 8x - 24$	yes
S_{440G}	$y^2 = x^5 - 2x^3 - 7x^2 - 8x + 8$	yes
S_{441I}	$y^2 = -3\,x^6 + 12\,x^4 + 6\,x^3 - 24\,x^2 - 36\,x - 27$	yes
S_{464I}	$y^2 = -x^6 - 2x^5 - 7x^4 - 6x^3 - 13x^2 - 4x - 8$	yes
S_{476B}	$y^2 = x^5 + 2x^4 + 3x^3 + 6x^2 + 4x + 1$	yes
S_{476D}	$y^2 = x^5 - 2x^4 + 3x^3 - 6x^2 - 7$	yes
S_{483C}	$y^2 = x^6 + 12\,x^5 + 26\,x^4 - 34\,x^3 - 67\,x^2 + 90\,x - 27$	yes
S_{488A}	$y^2 = -3x^6 + 18x^5 - 27x^4 - 12x^3 - 27x^2 - 36x - 24$	yes

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